Polynomial-Complexity Computation of the M-phase Vector that Maximizes a Rank-Deficient Quadratic Form

George N. Karystinos and Anastasios T. Kyrillidis
Department of Electronic and Computer Engineering
Technical University of Crete
Kounoupidiana, Chania, 73100, Greece
E-mail: $\{\text{karystinos}, \text{kyrillidis}\}$@telecom.tuc.gr

***Abstract*—The maximization of a positive (semi)definite complex quadratic form over a finite alphabet is $\mathcal{NP}$-hard and achieved through exhaustive search when the form has full rank. However, if the form is rank-deficient, the optimal solution can be computed with only polynomial complexity in the length N of the maximizing vector. In this work, we consider the general case of a rank-D positive (semi)definite complex quadratic form and develop a method that maximizes the form with respect to a M-phase vector with polynomial complexity. The proposed method efficiently reduces the size of the feasible set from exponential to polynomial. We also develop an algorithm that constructs the polynomial-size candidate set in polynomial time and observe that it is fully parallelizable and rank-scalable.**

***Index Terms*—Quadratic form maximization, M-PSK, Polynomial complexity**

I. Introduction

The problem of unconstrained complex quadratic maximization over a finite alphabet captures many problems that are of interest to the communications and signal processing community. Many recent developments on the semidefinite relaxation (SDR) technique have indicated that SDR is capable of providing near optimal (and sometimes accurate) approximations in polynomial time [1]. Although SDR algorithm has, at worst, moderate approximation accuracy [2], in most of the times it remains an approximation algorithm that does not guarantee the computation of the optimal solution.

Interestingly, it has been recently proven that the maximization of a quadratic form with a binary vector argument[1] is no longer $\mathcal{NP}$-hard if the rank of the form is not a function of the problem size [3] and can be computed optimally and efficiently in polynomial time through a variety of computational geometry (CG) algorithms, such as the incremental algorithm for cell enumeration in arrangements [4] and the reverse search method [5],[6]. However, it should be noted that, although the incremental algorithm is optimal and applicable to M-ary phase-shift keying (PSK) modulation with $M > 4$, it is not known whether has practical importance in higher dimensions due to lack of parallelizability and memory management inefficiency. On the other hand, the reverse search method is highly parallelizable, speed and memory efficient but can be applied only for binary phase-shift keying (BPSK) and quaternary phase-shift keying (QPSK) modulations.

[1]In this work, a vector is called binary if and only if each element of it equals $+1$ or -1. Contrarily, if each element of it equals 0 or 1, then the vector is said to belong to the $0/1$ field.

On the other hand, recent developments on various telecommunications problems led to algorithms that optimally solve the problem of maximization of a rank-D quadratic form over the M-PSK alphabet for low values of D. Specifically, the authors in [7] (see also references therein) devised a new lattice decoding algorithm for the efficient computation of the M-phase vector when the rank of the quadratic form satisfies $D \leq 2$; if $D > 2$, the proposed algorithm provides suboptimal solutions.

In the present work, we modify the algorithm in [8] to serve complex-domain optimization problems. More specifically, we present an algorithm for the efficient computation of the M-ary phase vector of length N that maximizes a rank-D quadratic form with polynomial complexity where $D < N$ and D independent of N. The algorithm uses $2D-1$ auxiliary hyperspherical coordinates that partition the multidimensional hypercube into distinct regions of polynomial size, each of which corresponds to a different candidate vector. Therefore, the method reduces the size of the candidate vector set from exponential to polynomial and the proposed algorithm turns out to be time and memory efficient, fully parallelizable and rank-scalable.

II. Problem Statement

We consider the quadratic expression[2]

$$\mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s} \tag{1}$$

where $\mathbf{Q} \in \mathbb{C}^{N\times N}$ is a positive (semi)definite matrix and $\mathbf{s} \in \mathcal{A}_M^N$ is a M-PSK N-tuple vector argument. We assume $\mathcal{A}_M = \left\{ e^{\frac{j2\pi m}{M}} \;\middle|\; m = 0, 1, \ldots, M-1 \right\}$ as the M-phase alphabet and $M \in \{2^k \mid k = 1, 2 \ldots\}$.

In our problem, we focus on the computation of the M-PSK vector that maximizes the quadratic form

$$\mathbf{s}_{\text{opt}} \triangleq \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}. \tag{2}$$

[2]**Notation:** Upper and lower case bold symbols denote matrices and column vectors, respectively; x_i denotes the i-th element of vector $\mathbf{x}$ and $A_{i,j}$ the (i,j)-th entry of matrix $\mathbf{A}$; $\mathbf{A}_{i:j,k:l}$ follows a MATLAB-like notation that denotes the submatrix of $\mathbf{A}$ that consists of the i-th up to j-th rows and k-th up to l-th columns of it; $(\cdot)^*$ denotes conjugation; $(\cdot)^T$ transpose; $(\cdot)^{\mathcal{H}}$ Hermitian transpose; $\mathbf{0}_{N\times 1}$ the $N \times 1$ vector of all zeros; $\|\cdot\|$ the Frobenius norm; $|\cdot|$ the cardinality of a set and $j \triangleq \sqrt{-1}$.

Since $\mathbf{Q}$ is symmetric, the matrix can be represented in terms of its eigenvalues and eigenvectors using spectral factorization

$$\mathbf{Q} = \sum_{n=1}^{N} \lambda_n \mathbf{q}_n \mathbf{q}_n^{\mathcal{H}}, \;\; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N, \;\; \mathbf{q}_n \in \mathbb{C}^N,$$
$$\|\mathbf{q}_n\| = 1, \;\; \mathbf{q}_n^{\mathcal{H}} \mathbf{q}_k = 0, \;\; n \neq k, \;\; n, k = 1, 2, \ldots, N, \quad (3)$$

where λ_n and $\mathbf{q}_n$ are the n-th eigenvalue and eigenvector of the matrix $\mathbf{Q}$, respectively. If $\lambda_N > 0$, then $\mathbf{Q}$ is full rank and our problem in (2) becomes $\mathcal{NP}$-hard where the computation of $\mathbf{s}_{\rm opt}$ can be implemented using exhaustive search over the set $\mathcal{A}_M^N$ with complexity[3] $\mathcal{O}(M^N)$ since $|\mathcal{A}_M^N| = M^N$.

On the other hand, if $\lambda_n = \lambda_{n+1} = \cdots = \lambda_N = 0, n \in \{2, 3, \ldots, N\}$, then $\mathbf{Q}$ is rank-deficient. Therefore, in the following, without loss of generality (w.l.o.g.), $\mathbf{Q}$ is assumed a positive (semi)definite complex matrix with rank $D \leq N$, i.e

$$\mathbf{Q} = \sum_{n=1}^{D} \lambda_n \mathbf{q}_n \mathbf{q}_n^{\mathcal{H}}, \; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_D > 0. \quad (4)$$

Furthermore, since $\lambda_n > 0, n = 1, 2, \ldots, D$, we define the weighted principal component

$$\mathbf{v}_n \triangleq \sqrt{\lambda_n} \mathbf{q}_n, \;\; n = 1, 2, \ldots, D, \quad (5)$$

and the corresponding $\mathbf{V} \in \mathbb{C}^{N \times D}$ complex matrix

$$\mathbf{V} \triangleq \begin{bmatrix} \mathbf{v}_1 \; \mathbf{v}_2 \; \ldots \mathbf{v}_D \end{bmatrix} \quad (6)$$

such that $\mathbf{V}\mathbf{V}^{\mathcal{H}} = \sum_{n=1}^{D} \mathbf{v}_n \mathbf{v}_n^{\mathcal{H}} = \sum_{n=1}^{D} \lambda_n \mathbf{q}_n \mathbf{q}_n^{\mathcal{H}} = \mathbf{Q}$. Thus, our initial problem statement in (2) can be transformed into the following optimization problem

$$\mathbf{s}_{\rm opt} \triangleq \arg \max_{\mathbf{s} \in \mathcal{A}_M^N} \left\{ \mathbf{s}^{\mathcal{H}} \mathbf{V} \mathbf{V}^{\mathcal{H}} \mathbf{s} \right\}. \quad (7)$$

We underline that $\mathbf{V}$ is a full rank complex matrix and matrices $\mathbf{Q}$ and $\mathbf{V}$ have the same rank $D \leq N$.

In the next section, we use the framework presented in [8] and propose a more generalized algorithm for the maximization of a rank-deficient quadratic form over any M-ary PSK alphabet where $M \in \{2^k \mid k = 1, 2 \ldots\}$.

III. Efficient Maximization of Rank-deficient Quadratic Form with a MPSK Vector Argument

A. Problem Reformulation

Since $\mathbf{s}^{\mathcal{H}} \mathbf{V} \mathbf{V}^{\mathcal{H}} \mathbf{s} = \|\mathbf{V}^{\mathcal{H}} \mathbf{s}\|^2$, we can rewrite our problem as

$$\mathbf{s}_{\rm opt} \triangleq \arg \max_{\mathbf{s} \in \mathcal{A}_M^N} \|\mathbf{V}^{\mathcal{H}} \mathbf{s}\|. \quad (8)$$

W.l.o.g., we assume that each row of $\mathbf{V}$ has at least one nonzero element, i.e. $\mathbf{V}_{n,1:D} \neq \mathbf{0}_{1 \times D}, \forall n \in \{1, 2, \ldots, N\}$. In opposite case the value of the maximization argument $s_n, n \in \{1, 2, \ldots, N\}$, related with the n-th all-zero row of $\mathbf{V}_{N \times D}$ would have no effect on the maximization. Therefore, assuming that we have $\mathcal{K} \in \{1, 2, \ldots N\}$ rows of $\mathbf{V}$ equal to $\mathbf{0}_{1 \times D}$, we can simply ignore these rows, reduce the dimension of our problem from N to $N - \mathcal{K}$ and assign arbitrary values to the elements of the maximizing vector related to the $\mathcal{K}$ all-zero rows of $\mathbf{V}$.

Let $\boldsymbol{\phi}_{i:j} \triangleq [\phi_i, \phi_{i+1}, \ldots, \phi_j]^T$. To develop an efficient method for the maximization in (8), we introduce $2D-1$ auxiliary hyperspherical coordinates $\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ and define the hyperspherical real vector with unit radial coordinate,

$$\tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1}) \triangleq \begin{bmatrix} \sin \phi_1 \\ \cos \phi_1 \sin \phi_2 \\ \cos \phi_1 \cos \phi_2 \sin \phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-2} \cos \phi_i \right] \sin \phi_{2D-1} \\ \left[\prod_{i=1}^{2D-2} \cos \phi_i \right] \cos \phi_{2D-1} \end{bmatrix}_{2D \times 1} \quad (9)$$

as well as the $D \times 1$ hyperspherical complex vector $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) \triangleq \tilde{\mathbf{c}}_{2:2:2D}(\boldsymbol{\phi}_{1:2D-1}) + j \tilde{\mathbf{c}}_{1:2:2D-1}(\boldsymbol{\phi}_{1:2D-1})$.

From Cauchy-Swartz inequality, we observe that for any $\mathbf{a} \in \mathbb{C}^D$,

$$\left| \mathbf{a}^{\mathcal{H}} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) \right| \leq \|\mathbf{a}\| \underbrace{\|\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\|}_{=1} = \|\mathbf{a}\|. \quad (10)$$

The equality of (10) is achieved if and only if $\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ are the hyperspherical coordinates of vector $\mathbf{a}$, i.e. if [4]

$$\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad (11)$$

since $\left| \mathbf{a}^{\mathcal{H}} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) \right| = \left| \mathbf{a}^{\mathcal{H}} \frac{\mathbf{a}}{\|\mathbf{a}\|} \right| = \|\mathbf{a}\|$. Using the above, a critical equality for our subsequent developments is

$$\begin{aligned} \mathbf{s}_{\rm opt} &= \arg \max_{\mathbf{s} \in \mathcal{A}_M^N} \|\mathbf{V}^{\mathcal{H}} \mathbf{s}\| \\ &= \arg \max_{\mathbf{s} \in \mathcal{A}_M^N} \max_{\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]} \left| \mathbf{s}^{\mathcal{H}} \mathbf{V} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) \right|. \end{aligned} \quad (12)$$

Furthermore, we observe that for any $\mathbf{a} \in \mathbb{C}^D$ and any $\hat{\theta} \in (-\pi, \pi]$,

$$\Re\left\{ \mathbf{a}^{\mathcal{H}} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) e^{-j\hat{\theta}} \right\} \leq \left| \mathbf{a}^{\mathcal{H}} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) \right| \quad (13)$$

with equality if and only if $\hat{\theta} = \arg \left\{ \mathbf{a}^{\mathcal{H}} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) \right\}$.

It can be easily observed that expressions (10) and (13) are simultaneously satisfied with equality if and only if $\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ are the hyperspherical coordinates of

[3]Since rotated candidate vectors $\hat{\mathbf{s}} = \mathbf{s} e^{j \frac{2\pi m}{M}}, m = 0, 1, \ldots, M-1$, give the same result in our maximization problem, i.e. $\hat{\mathbf{s}}^{\mathcal{H}} \mathbf{Q} \hat{\mathbf{s}} = (\mathbf{s} e^{j \frac{2\pi m}{M}})^{\mathcal{H}} \mathbf{Q} (\mathbf{s} e^{j \frac{2\pi m}{M}}) = \mathbf{s}^{\mathcal{H}} \mathbf{Q} \mathbf{s}$, we can focus only on the $\frac{1}{M}$-th of the elements of $\mathcal{A}_M^N$. In this case, the complexity of the resulting maximization quadratic form reduces to $\mathcal{O}(M^{N-1})$, which is still intractable for moderate values of N.

[4]We observe that the equality of (10) is also achieved for any rotated version of $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})$, i.e. $e^{j\omega} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = e^{j\omega} \frac{\mathbf{a}}{\|\mathbf{a}\|}$ for any $\omega \in (-\pi, \pi]$ since $\left| \mathbf{a}^{\mathcal{H}} e^{j\omega} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) \right| = \left| \mathbf{a}^{\mathcal{H}} e^{j\omega} \frac{\mathbf{a}}{\|\mathbf{a}\|} \right| = \left| e^{j\omega} \|\mathbf{a}\| \right| = \|\mathbf{a}\|$. But, for clarity reasons and w.l.o.g., we present the case for $\omega = 0$ in the above statement.

vector $\mathbf{a}$ ***and*** $\hat{\theta}=\theta$. Then, applying some computations, (12) can be further transformed into:

$$\mathbf{s}_{\rm opt}=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\max_{\substack{\boldsymbol{\phi}_{1:2D-1}\in\\(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\pi,\pi]}}\Re\Big\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\Big\}. \tag{14}$$

Next, we note the following: given a hyperspherical complex vector $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})$ and $\phi_{2D-1}\in(-\pi,\pi]$, there always exists an angle $\alpha\in\arg\{\mathcal{A}_M\}$ where $\arg\{\mathcal{A}_M\}=\Big\{\frac{2\pi m}{M}\mid m=0,1,\ldots,M-1\Big\}$ that relocates the angular coordinate ϕ_{2D-1} of the hyperspherical vector $\Big\{\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{j\alpha}\Big\}$ in the interval $(-\frac{\pi}{M},\frac{\pi}{M}]$ and results to the same value in the metric of (14).

Thus, without loss of optimality, we choose $\alpha\in\arg\{\mathcal{A}_M\}$ such that $\phi_{2D-1}\in(-\frac{\pi}{M},\frac{\pi}{M}]$. Thus, (14) becomes

$$\mathbf{s}_{\rm opt}=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\max_{\substack{\boldsymbol{\phi}_{1:2D-1}\in\\(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]}}\Re\Big\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\Big\}. \tag{15}$$

Dropping the arg operator and interchanging the maximizations in (15) we obtain the equivalent problem where $\Phi\triangleq\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big]$

$$\max_{\boldsymbol{\phi}_{1:2D-1}\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]}\sum_{n=1}^{N}\max_{s_n\in\mathcal{A}_M}\Re\Big\{s_n^*\mathbf{V}_{n,1:D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\Big\}. \tag{16}$$

B. Decision Functions and Candidate Vector Set $\mathcal{S}(\mathbf{V}_{N\times D})$

We observe that the original maximization problem in (8) is decomposed in a set of symbol-by-symbol maximization rules for a given set of angles $\boldsymbol{\phi}_{1:2D-1}\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$. For such a set of angles, the maximization argument of the sum in (16), e.g. symbol s_n, depends only on the corresponding row of matrix $\mathbf{V}$. As $\boldsymbol{\phi}_{1:2D-1}$ vary, the decision in favor of s_n is maintained as long as a decision boundary is not crossed.

Due to the structure of $\mathcal{A}_M$ and given the definitions above, the $\frac{M}{2}$ decision boundaries for the determination of s_n are given by

$$\begin{aligned}\mathbf{V}_{n,1:D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) =&Ae^{j\pi\frac{2k+1}{M}},\\ &A\in\mathbb{R},\ \ k=0,1,\ldots,\frac{M}{2}-1,\end{aligned} \tag{17}$$

or equivalently

$$\Im\Big\{e^{-j\pi\frac{2k+1}{M}}\mathbf{V}_{n,1:D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\Big\}=0,\ \ k=0,1,\ldots,\frac{M}{2}-1. \tag{18}$$

For $n=1,2,\ldots,N$ and $k=0,1,\ldots,\frac{M}{2}-1$, we can rewrite (18) as

$$\tilde{\mathbf{V}}_{l,1:2D}\tilde{\mathbf{c}}(\phi_{1:2D-1})=0,\ \ l=1,2,\ldots,\frac{MN}{2} \tag{19}$$

where

$$\tilde{\mathbf{V}}_{:,1:2:2D-1}=\Re(\hat{\mathbf{V}})\quad\text{and}\quad\tilde{\mathbf{V}}_{:,2:2:2\mathrm{D}}=\Im(\hat{\mathbf{V}}) \tag{20}$$

with $\hat{\mathbf{V}}=\mathbf{V}\otimes[e^{j\frac{\pi}{M}}\ e^{j\frac{3\pi}{M}}\ \cdots\ e^{j\frac{(M-1)\pi}{M}}]^T$ and $\otimes$ denotes the Kronecker product.

Motivated by the statements above and the inner maximization rule in (16), for each $D\times 1$ complex vector $\mathbf{v}$ we define the decision function s that maps $\boldsymbol{\phi}_{1:2D-1}$ to $\mathcal{A}_M$ according to

$$s(\mathbf{v}^T;\boldsymbol{\phi}_{1:2D-1})\triangleq\arg\max_{s\in\mathcal{A}_M}\Re\{s^*\mathbf{v}^T\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\}. \tag{21}$$

Furthermore, for the given $N\times D$ complex observation matrix $\mathbf{V}$, we can construct the vector decision function $\mathbf{s}$ using (21) where each point $\boldsymbol{\phi}_{1:2D-1}\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ is mapped to a candidate M-PSK vector according to

$$\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-1})\triangleq\begin{bmatrix}s(\mathbf{V}_{1,1:D};\boldsymbol{\phi}_{1:2D-1})\\ s(\mathbf{V}_{2,1:D};\boldsymbol{\phi}_{1:2D-1})\\ \vdots\\ s(\mathbf{V}_{N,1:D};\boldsymbol{\phi}_{1:2D-1})\end{bmatrix} \tag{22}$$

Computing $\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-1})$ for $\forall\boldsymbol{\phi}_{1:2D-1}\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$, we collect all M-phase candidate vectors into a set

$$\mathcal{S}(\mathbf{V}_{N\times D})\triangleq\bigcup_{\substack{\boldsymbol{\phi}_{1:2D-1}\in\\ \Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]}}\Big\{\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-1})\Big\}\subseteq\mathcal{A}_M^N. \tag{23}$$

Since $\boldsymbol{\phi}_{1:2D-1}$ take values from the set $\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$, our problem in (7) becomes

$$\mathbf{s}_{\rm opt}\triangleq\arg\max_{\mathbf{s}\in\mathcal{S}(\mathbf{V})}\Big\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{V}^{\mathcal{H}}\mathbf{s}\Big\}, \tag{24}$$

i.e. the M-phase candidate vector $\mathbf{s}_{\rm opt}$ that maximizes the expression above belongs into the set $\mathcal{S}(\mathbf{V}_{N\times D})$.

In the following, we (i) show that $|\mathcal{S}(\mathbf{V}_{N\times D})|=\sum_{d=1}^{D}\sum_{i=0}^{d-1}\binom{N}{i}\binom{N-i}{2(d-i)-1}\left(\frac{M}{2}\right)^{2(d-i)-2}\left(\frac{M}{2}-1\right)^{i}$ and (ii) develop an algorithm for the construction of $\mathcal{S}(\mathbf{V}_{N\times D})$ with complexity $\mathcal{O}((\frac{MN}{2})^{2D})$.

C. Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$

According to eq. (19), the rows of $\tilde{\mathbf{V}}_{\frac{MN}{2}\times 2D}$ determine $\frac{MN}{2}$ hypersurfaces $\boldsymbol{\mathcal{H}}\triangleq\{\mathcal{H}(\tilde{\mathbf{V}}_{1,1:2D}),\mathcal{H}(\tilde{\mathbf{V}}_{2,1:2D}),\ldots,\mathcal{H}(\tilde{\mathbf{V}}_{\frac{MN}{2},1:2D})\}$ that partition the $(2D-1)$-dimensional hypercube $\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ into K non-interleaving cells $C_1,C_2,\ldots,C_K$ such that the union of all cells is equal to $\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ and the intersection of any two distinct cells, say C_k,C_j for $k\neq j$, is empty. Each cell C_k corresponds to a *distinct* $\mathbf{s}_k\in\mathcal{A}_M^N$ in the sense that $\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-1})=\mathbf{s}_k$ for any $\boldsymbol{\phi}_{1:2D-1}\in C_k$ and $\mathbf{s}_k\neq\mathbf{s}_j$ if $k\neq j, k,j\in\{1,2,\ldots,K\}$.

Let $\mathcal{I}_{2D-1}\triangleq\{i_1,i_2,\ldots,i_{2D-1}\}\subset\{1,2,\ldots,\frac{MN}{2}\}$ denote the subset of $2D-1$ indices that correspond to hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2D}),\mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2D}),\ldots,\ \mathcal{H}(\tilde{\mathbf{V}}_{i_{2D-1},1:2D})$. We detect the following cases:

(a) Intersections of $2D-1$ hypersurfaces where at most two surfaces originate from the same row of $\mathbf{V}$.

(b) Intersections of $2D-1$ hypersurfaces where at least three surfaces originate from the same row of $\mathbf{V}$.

Two basic properties of such intersections are presented in the following proposition.

Proposition 2: The following hold true.

(i) Each subset of $\boldsymbol{\mathcal{H}}$ that consists of $2D-1$ hypersurfaces has either a single or uncountably many intersections in $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$.

(ii) Each combination of $2D-1$ hypersurfaces from the set $\boldsymbol{\mathcal{H}}$ has a unique intersection point that constitutes a vertex of a cell if and only if no more than two hypersurfaces originate from the same row of the matrix $\mathbf{V}$. ■

According to Proposition 2, Part (ii), combinations of the form (b) do not have a unique intersection point but infinitely many intersection points; thus no cell is created and these combinations can be ignored.

On the other hand, combinations of the form (a) have a unique intersection point $\boldsymbol{\phi}(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1}) \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ that leads $\mathcal{Q}$ cells, say $C_1(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})$, $C_2(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})$, $\ldots$, $C_{\mathcal{Q}}(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})$, $\mathcal{Q} \in \left\{(\frac{M}{2}-1)^0, (\frac{M}{2}-1)^1, \ldots, (\frac{M}{2}-1)^{D-1}\right\}$ and each cell is associated with a distinct M-phase candidate vector $\mathbf{s}_q(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})$, $q = 1, 2, \ldots, \mathcal{Q}$, in the sense that $\mathbf{s}_q(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{2D-1}) = \mathbf{s}_q(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})$ for all $\boldsymbol{\phi}_{1:2D-1} \in C_q(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})$ and $\boldsymbol{\phi}(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})$ is a single point of $C_q(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})$ where ϕ_{2D-1} is minimized. The number of cells $\mathcal{Q}$ "led" by an intersection point depends on the number p of participating pairs of hypersurfaces that originate from the same row of matrix $\mathbf{V}$ and equals to $(\frac{M}{2}-1)^p$.

Since each cell is associated with a distinct M-PSK candidate vector, we can collect all these vectors into

$$\mathcal{J}(\mathbf{V}_{N\times D}) \triangleq \bigcup_{\substack{\mathcal{I}_{2D-1}\subset\{1,2,\ldots,\frac{MN}{2}\},\\ \boldsymbol{\phi}(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})\in\\ \Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]}} \left\{\mathbf{s}(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})\right\} \subseteq \mathcal{A}_M^N. \tag{25}$$

Taking into consideration only cells into the region of interest $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$, we observe that $|\mathcal{J}(\mathbf{V}_{N\times D})| = \sum_{i=0}^{D-1} \binom{N}{i}\binom{N-i}{2(D-i)-1} \frac{M}{2}^{2(D-i)-2}\left(\frac{M}{2}-1\right)^i$, i.e. there are $|\mathcal{J}(\mathbf{V}_{N\times D})|$ candidate vectors $\mathbf{s}$ in $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$, associated with cells each of which minimizes ϕ_{2D-1} component at a single point that constitutes the intersection of the corresponding $2D-1$ hypersurfaces. Additionally, it can be shown that if we take into consideration all regions in $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$, all candidates form the candidate set $\mathcal{S}$ given by

$$\begin{aligned}\mathcal{S}(\mathbf{V}_{N\times D}) &= \mathcal{J}(\mathbf{V}_{N\times D}) \cup \mathcal{J}(\mathbf{V}_{N\times(D-1)}) \cup \cdots \cup \mathcal{J}(\mathbf{V}_{N\times 1}) \\ &= \bigcup_{d=0}^{D-1} \mathcal{J}(\mathbf{V}_{N\times(D-d)}).\end{aligned} \tag{26}$$

with cardinality

$$\begin{aligned}&|\mathcal{S}(\mathbf{V}_{N\times D})| \\ &\quad= |\mathcal{J}(\mathbf{V}_{N\times D})| + \cdots + |\mathcal{J}(\mathbf{V}_{N\times 1})| \\ &\quad= \sum_{d=1}^{D}\sum_{i=0}^{d-1} \binom{N}{i}\binom{N-i}{2(d-i)-1}\left(\frac{M}{2}\right)^{2(d-i)-2}\left(\frac{M}{2}-1\right)^i \\ &\quad= \mathcal{O}\left(\left(\frac{MN}{2}\right)^{2D-1}\right).\end{aligned} \tag{27}$$

To summarize the results, we have partitioned the hypercube $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ into a finite number of cells $\mathcal{S}(\mathbf{V}_{N\times D})$ that are associated with distinct M-phase vectors and proved that $\mathbf{s}_{\rm opt} \in \mathcal{S}(\mathbf{V}_{N\times D})$. Therefore, the initial problem in (8) has been converted into numerical maximization of $\|\mathbf{V}^{\mathcal{H}}\mathbf{s}\|$ among all vectors $\mathbf{s} \in \mathcal{S}(\mathbf{V}_{N\times D})$.

IV. Algorithmic Developments and Complexity Study

In this section, we present the basic steps of the proposed algorithm for the construction of $\mathcal{S}(\mathbf{V}_{N\times D})$ for arbitrary $N, D \in \mathbb{N}, D < N$ and $M \in \{2^k \mid k = 1, 2 \ldots\}$. From eq. (26), we observe that the initial problem of the determination of $\mathcal{S}(\mathbf{V}_{N\times D})$ can be divided into smaller parallel construction problems of $\mathcal{J}(\mathbf{V}_{N\times d})$ for $d = 1, \ldots, D$. Moreover, the construction of $\mathcal{J}(\mathbf{V}_{N\times d)})$ can be fully parallelized since the candidate vector(s) $\mathbf{s}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ can be computed independently for each $\mathcal{I}_{2d-1}$.

For the following statements, we assume a certain value for $d \in \{1, 2, \ldots, D\}$ and a certain set of indices $\mathcal{I}_{2d-1} = \{i_1, i_2, \ldots, i_{2d-1}\}$. According to the derivations in the previous section, the combination of hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2d}), \mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2d}), \ldots, \mathcal{H}(\tilde{\mathbf{V}}_{i_{2d-1},1:2d})$ intersects at a single point $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ that "leads" $\mathcal{Q}$ cells associated with $\mathcal{Q}$ different M-phase candidate vectors $\mathbf{s}_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$, $q = 1, 2, \ldots, \mathcal{Q}$. It can be shown that the evaluation of the decision function in (21) at the intersection of the $2D-1$ hypersurfaces under consideration determines definitely the corresponding symbol s_n if and only if no hypersurface originates from $\mathbf{V}_{n,1:d}$. For the hypersurfaces that pass through the intersection, the rule in (21) becomes ambiguous. In such a case, we have constructed dissambiguation rules that solve the ambiguity in polynomial time with respect to the length N.

The algorithm visits independently $|\mathcal{S}(\mathbf{V}_{N\times D})| = \mathcal{O}\big((\frac{MN}{2})^{2D-1}\big)$ intersections and computes the candidate M-phase vector(s) associated with each intersection. For each $\mathcal{I}_{2d-1}$, the cost of the algorithm is $\mathcal{O}\big(\frac{MN}{2}\big)$. Therefore the overall complexity of the algorithm for the computation of $\mathcal{S}(\mathbf{V}_{N\times D})$ with fixed $D < N$ becomes $\mathcal{O}\big((\frac{MN}{2})^{2D-1}\big)\mathcal{O}\big(\frac{MN}{2}\big) = \mathcal{O}\big((\frac{MN}{2})^{2D}\big)$.

We observe that the computation of the candidate vectors of $\mathcal{S}(\mathbf{V}_{N\times D})$ is performed independently from cell to cell, which implies that there is no need to store the data that have been used for each candidate and we only have to store the "best" vector that has been met. Therefore, the proposed method is fully parallelizable and its memory utilization is efficiently minimized, in constrast to the incremental algorithm in [4]. We also mention that if the initial problem is of a high rank that makes the optimization intractable, then the matrix $\mathbf{Q}$ in (2) can be approximated by keeping the D strongest principal components of it. In such a case, as seen in (26), the proposed method is rank-scalable.

Compared to previous works on the maximization of a complex rank-deficient quadratic form over a finite field, we recall that the reverse search method [5],[6] computes $\sum_{i=0}^{D-1}\binom{N-1}{i}$ candidates for the BPSK case (as many as our proposed algorithm) and $\sum_{i=0}^{2D-1}\binom{2N-1}{i}$ candidates for the QPSK case (twice as many as our proposed algorithm) [see Fig.1]. Additionally, the corresponding complexity of the algorithm proposed in [5],[6] is of the order $\mathcal{O}(N^{2D}\mathbf{LP}(\frac{MN}{2},2D))$ and $\mathcal{O}\big((2N)^{2D}\mathbf{LP}(\frac{MN}{2},2D)\big)$ for BPSK and QPSK respectively, where $\mathbf{LP}(\frac{MN}{2},2D)$ is the time to solve a linear programming ($\mathbf{LP}$) optimization problem with $\frac{MN}{2}$ inequalities in $2D$ variables. Given that the complexity of $\mathbf{LP}(\frac{MN}{2},2D)$ is linear in $\frac{MN}{2}$ in the worst-case scenario, it turns out that the complexity of reverse search method is $\mathcal{O}(\frac{MN}{2}^{2D+1})$ for $M=2,4$, i.e. one order of magnitude more calculations than the proposed algorithm. In addition, the reverse search method is restricted only to BPSK and QPSK modulation.

On the other hand, the incremental algorithm proposed in [4],[9] is a time efficient algorithm that solves the maximization problem of interest but becomes impractical even for moderate values of D since it follows an "incremental" strategy to construct the candidate set: it solves the problem inductively and, thus, it is too complicated to be implemented. Furthermore, the critical disadvantage of this method is its memory inefficiency since it needs to store all the extreme points, all faces and their incidences in memory. Finally, the algorithm proposed in [7] deals optimally the problem of the maximization of a rank-deficient quadratic form for any $M=\{2^k|k=0,1,\dots\}$ but only for $D\leq 2$. For $D>2$, the algorithm in [7] provides suboptimal solutions.

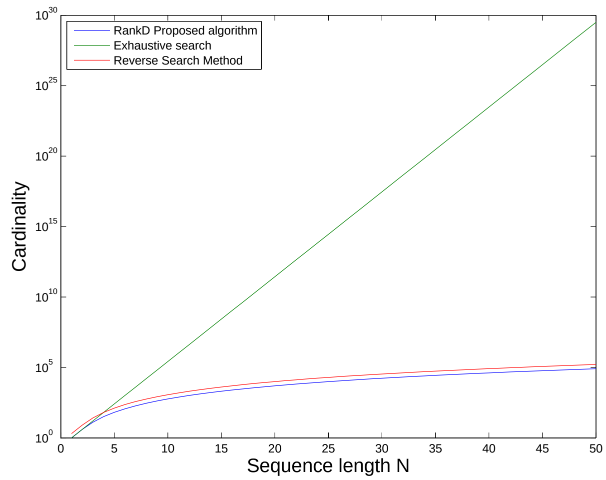

Fig. 1. Cardinality of candidate set $\mathcal{S}(\mathbf{V_{N\times D}})$ of proposed algorithm (blue line), reverse search method (red line) and exhaustive search (green line) - $D=2, M=4$.

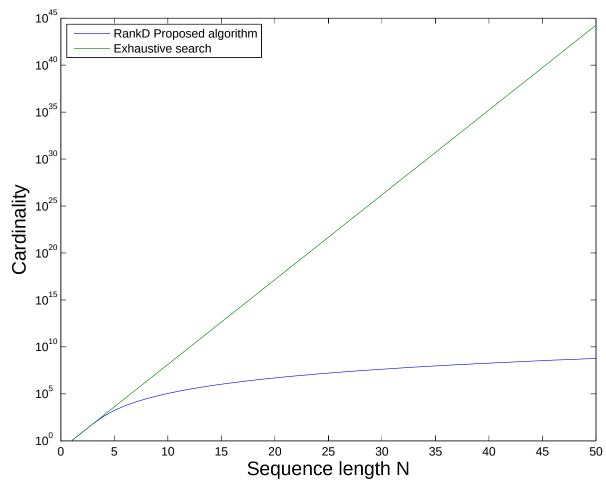

Fig. 2. Cardinality of candidate set $\mathcal{S}(\mathbf{V_{N\times D}})$ of proposed algorithm (blue line) and exhaustive search (green line) - $D=3, M=8$.

V. Conclusion

In this paper, we presented a more generalized, fully parallelizable, rank-scalable, time- and memory-efficient algorithm for the computation of the maximizing argument of a rank-deficient quadratic form over any M-ary PSK alphabet $\mathcal{A}_M^N$ in polynomial time in the length N of the maximizing argument. Thus, without loss of optimality, the proposed algorithm serves as an efficient alternative approach to exhaustive search for the computation of the maximizing M-ary phase vector $\mathbf{s}$ in the quadratic form $\mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}$.

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