COMP 414/514: Optimization – Algorithms, Complexity and Approximations

Lecture 8

Overview

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    - Talked a little bit about **general smooth optimization** problems
    - This included both non-convex optimization, but also convex
    - The discussion was quite abstract (no particular application)

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  - Talked a little bit about **general smooth optimization** problems
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- We will consider (possibly) the simplest non-convex setting: **sparse model selection**
  - We will provide motivation, background and alternative solutions
  - We will focus on how we can **provably and efficiently solve** such problems

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$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & x \in \mathcal{C} \end{aligned}$$

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$$\text{s.t.} \quad \boxed{x \in \mathcal{C}}$$

– We will focus on the cases of (structured) sparsity and low−rankness

Sparse linear regression

- Generative model: $y_i = a_i^\top x^\star + w_i$
  - $a_i \in \mathbb{R}^p$: features
  - $y_i \in \mathbb{R}$: responses
  - $w_i \in \mathbb{R}$: additive noise

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- Assuming data set $\{y_i, a_i\}_{i=1}^n$, $n < p$, find $x^\star \in \mathbb{R}^p$

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- Any suggestions how to solve this?

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– Solution #1: convexification + proj. gradient descent

LASSO

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$$\longrightarrow \quad x_{t+1} = \Pi_{\|\cdot\|_1 \leq \lambda}\left(x_t - \eta \nabla f(x_t)\right)$$

(Pros & Cons?)

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Basis pursuit (denoising)

– Solution #2: convexification + **proximal** gradient descent

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(Pros & Cons?)

– Solution #3: keep non−convexity + **non−convex projected gradient descent**

Hard-thresholding

$$\min_{x\in\mathbb{R}^p} \quad f(x) := \frac{1}{2}\|y - Ax\|_2^2 \qquad \text{s.t.} \quad \|x\|_0 \leq k \quad \longrightarrow \quad x_{t+1} = H_k\left(x_t - \eta\nabla f(x_t)\right)$$

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But before we proceed..

$$\min_{x \in \mathbb{R}^p} \quad f(x) := \frac{1}{2}\|y - Ax\|_2^2$$
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  - Q: ``How easy it is to solve $\ell_0$−pseudo norm problems?``
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(This assumes the most general case)

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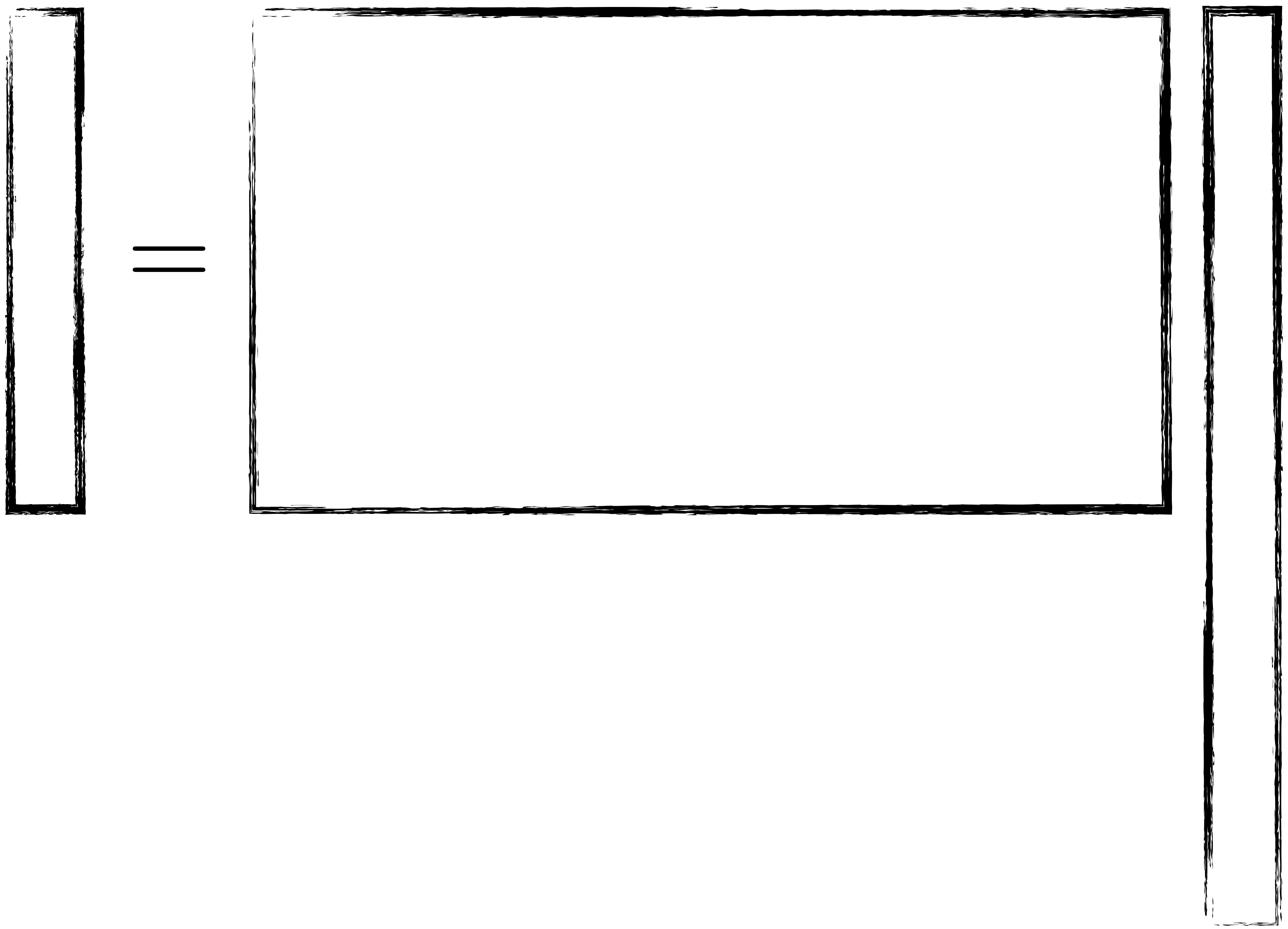
  - A: "Sparsity makes problems exponentially hard to solve"
    (This assumes the most general case)
  - Q: "But isn't the problem underdetermined? $(n \ll p)$"
  - A: "Yes, without any constraints, the problem has infinite solutions"
  - Q: "Why then do we have hopes solving this problem?"
  - A: "Under assumptions on A, and the relation between (n, p, k), we Will see that on average this problem can be solved in polynomial time"

Over- vs. under-parameterized



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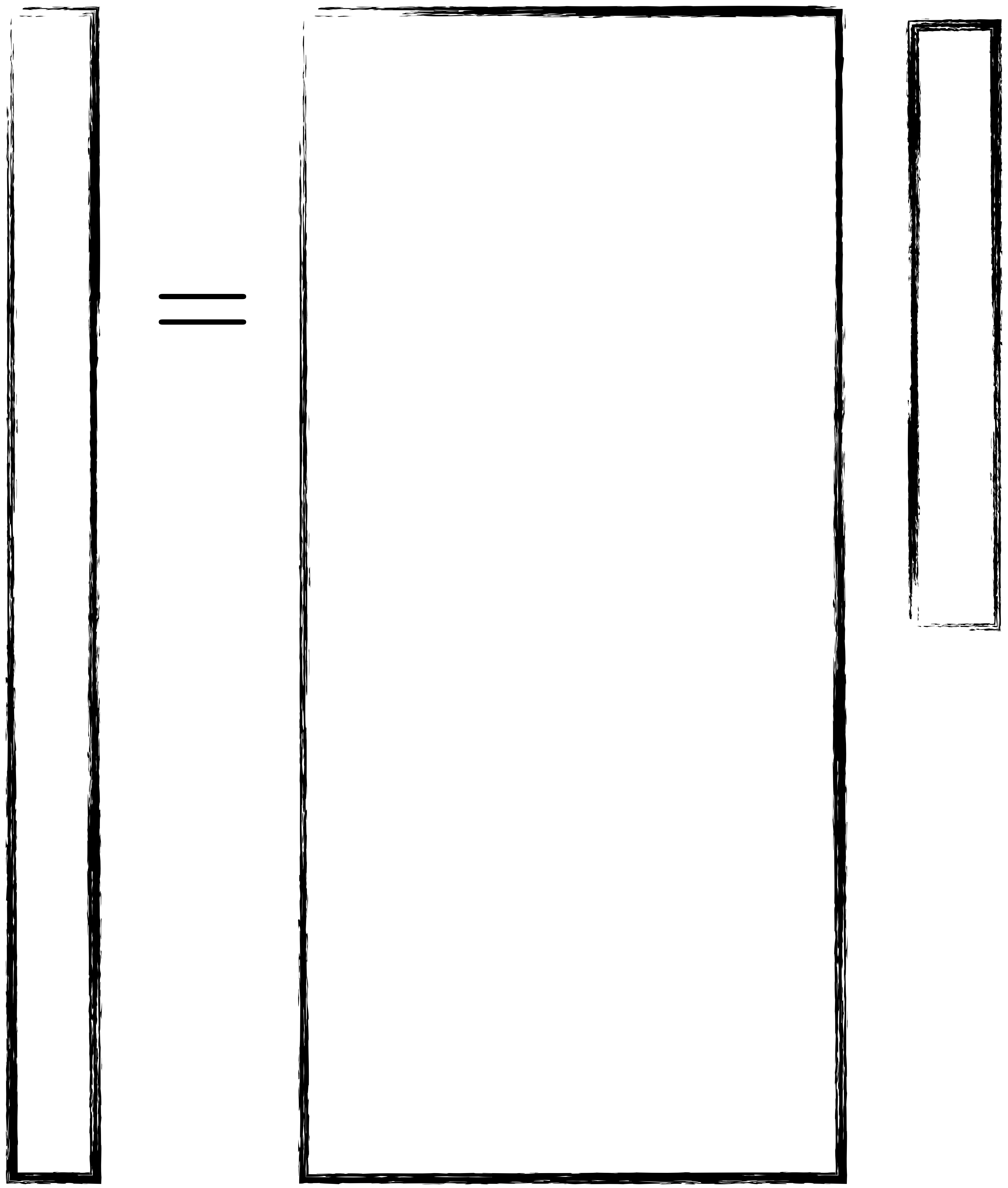

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- Now, imagine yourself implementing this.. What are the hyper−parameters?
  - "How do we set the step size?"
  - "How do we select the initial point? (it is non−convex after all)"
  - "What if we don't know the sparsity level?"
  - "Are there any other tricks we can use?"

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- Property of I: isometry

$$(1-\delta)\|x_1 - x_2\|_2^2 \leq \|I(x_1 - x_2)\|_2^2 \leq (1-\delta)\|x_1 - x_2\|_2^2, \quad \text{for some } \delta \in [0,1], \ \forall x_1, x_2 \in \mathbb{R}^p$$

(Interpretation?)

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- How can we use this property in proving convergence of IHT?

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- We get linear convergence to the global optimum!

How does it perform in practice?

Demo

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(In our case, we generate it as Gaussian so with high probability we are fine)

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Note: Checking whether a fixed matrix actually satisfies RIP is NP−hard..

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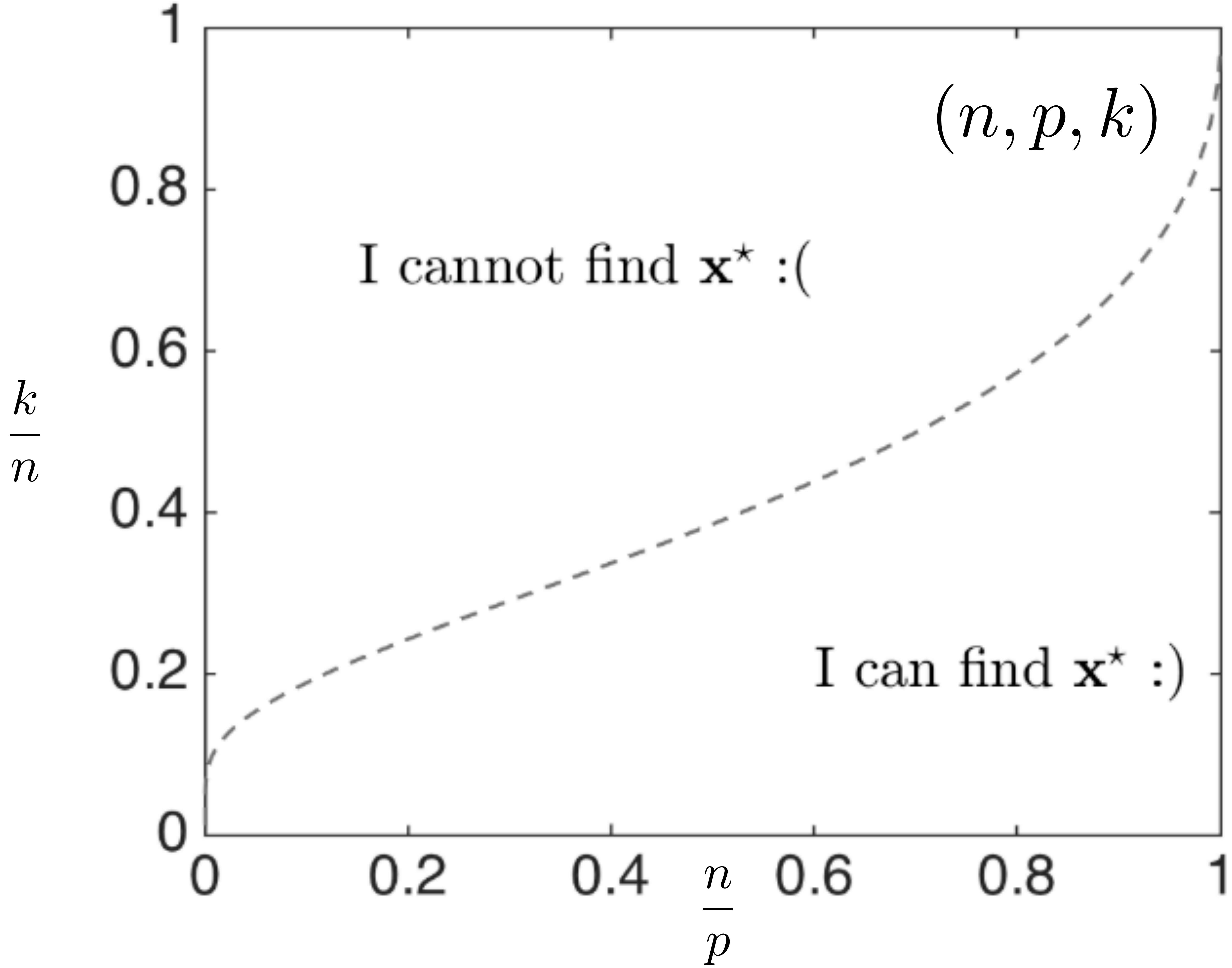

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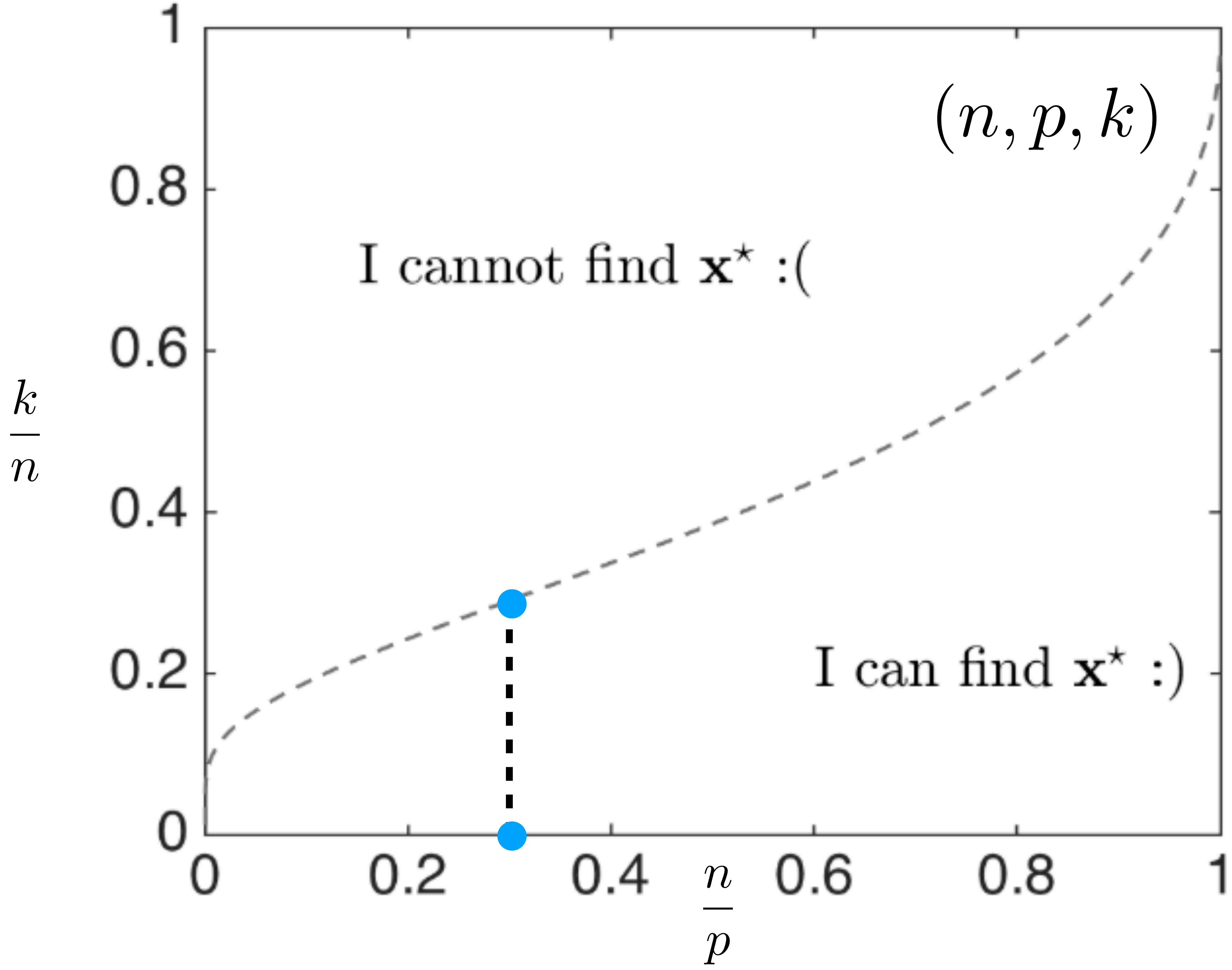

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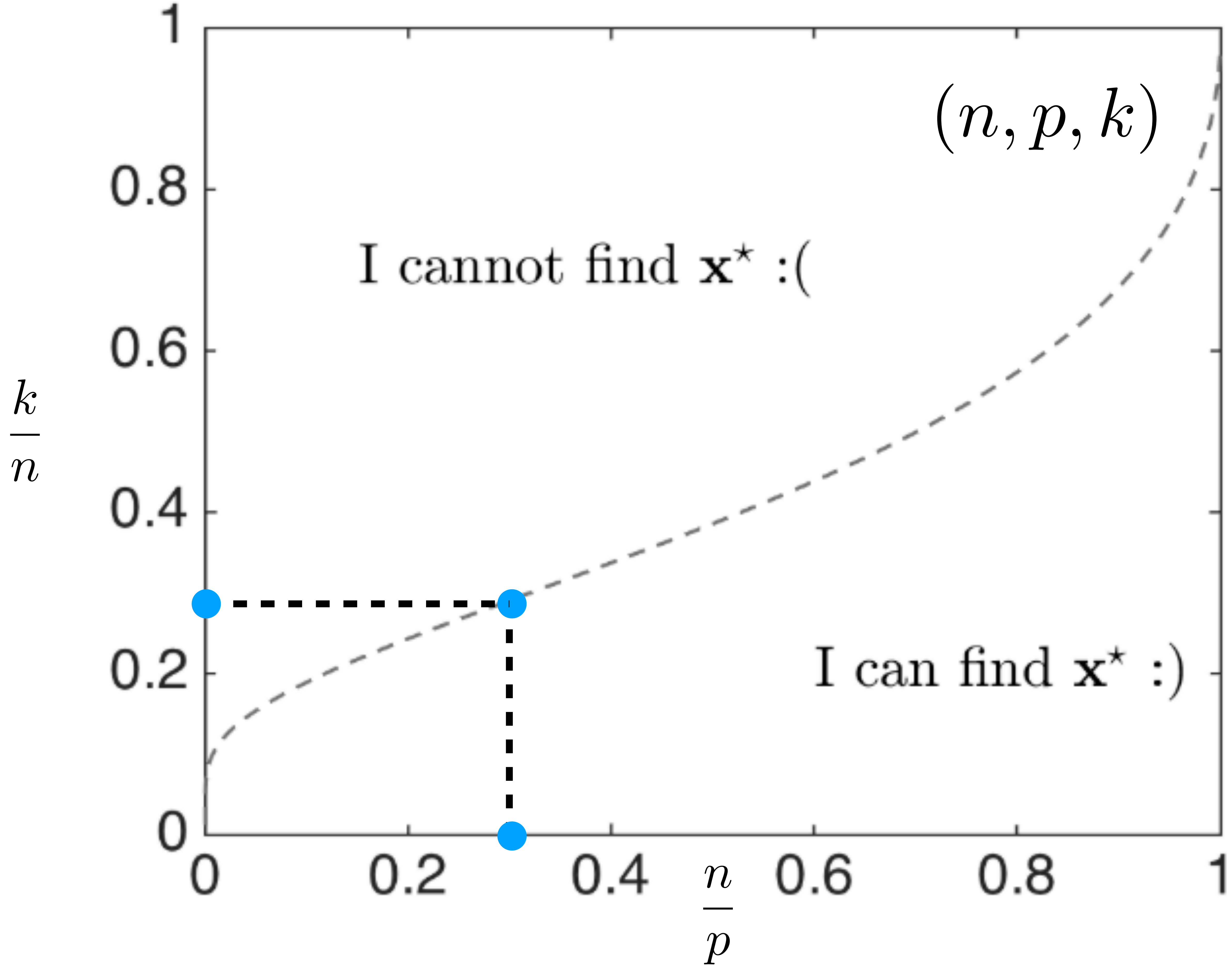

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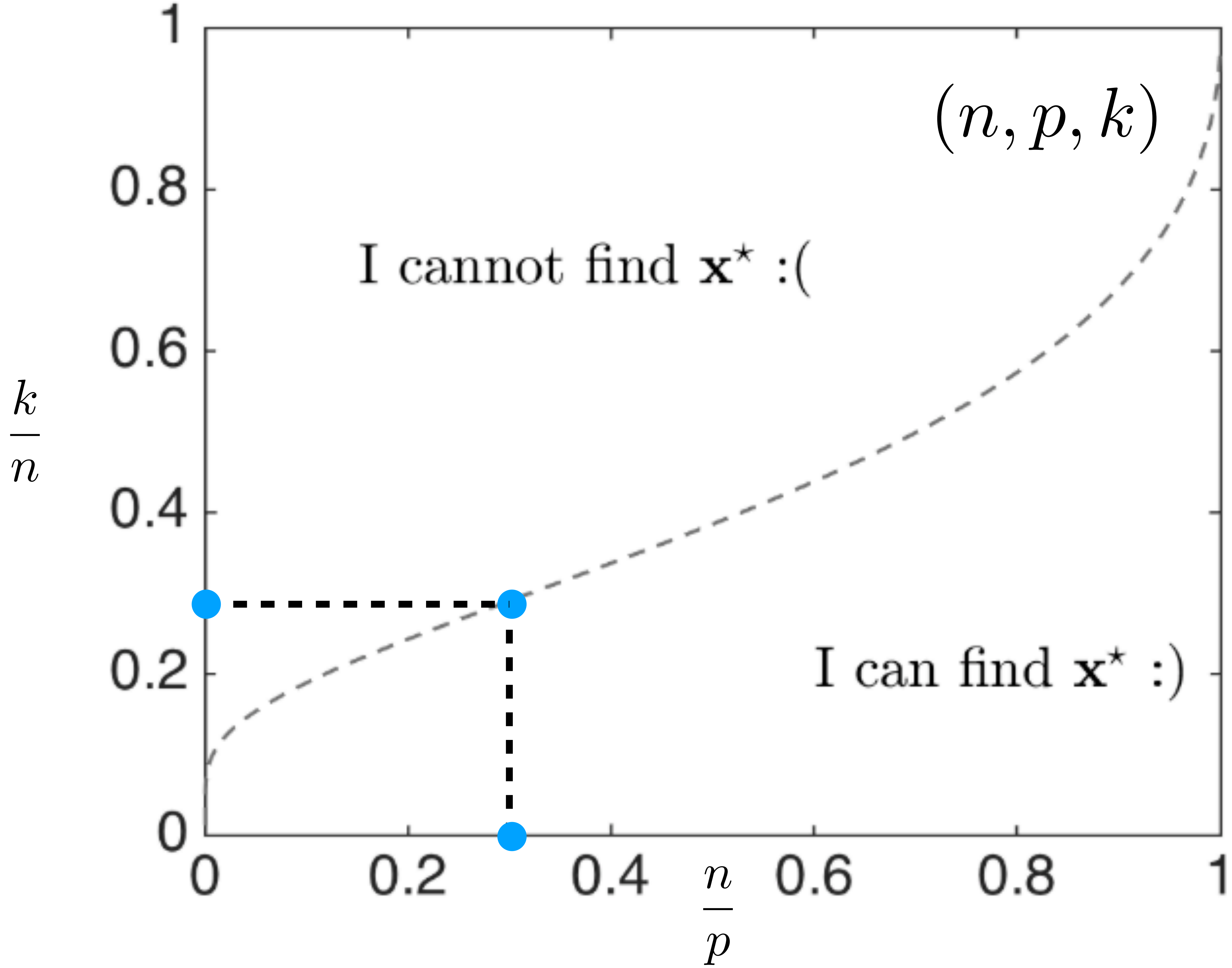

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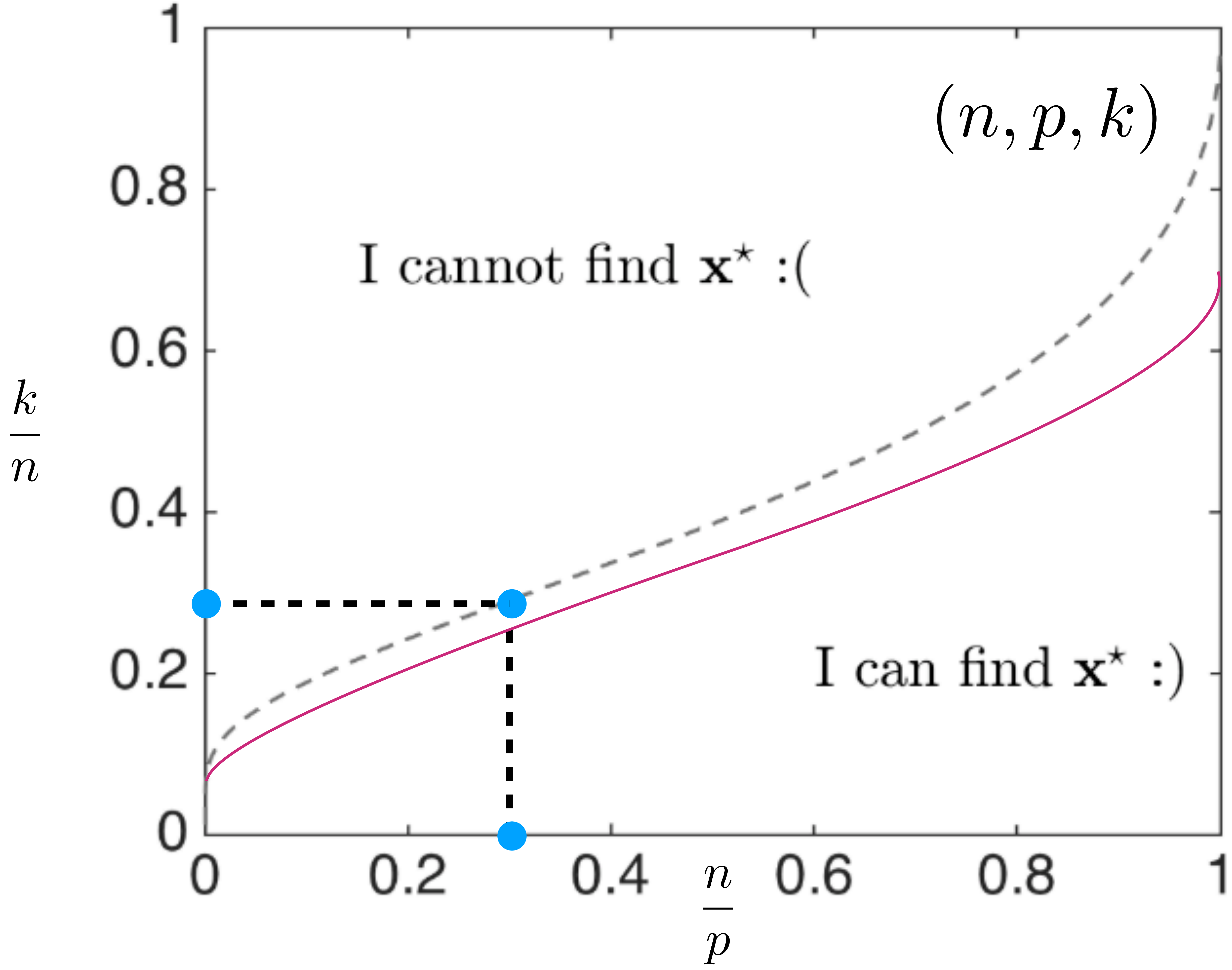

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Whiteboard

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(In literature section, there is work that validates theoretically this selection)

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- But, is this practical? Generally, no!

(Value of δ is NP−hard to find)

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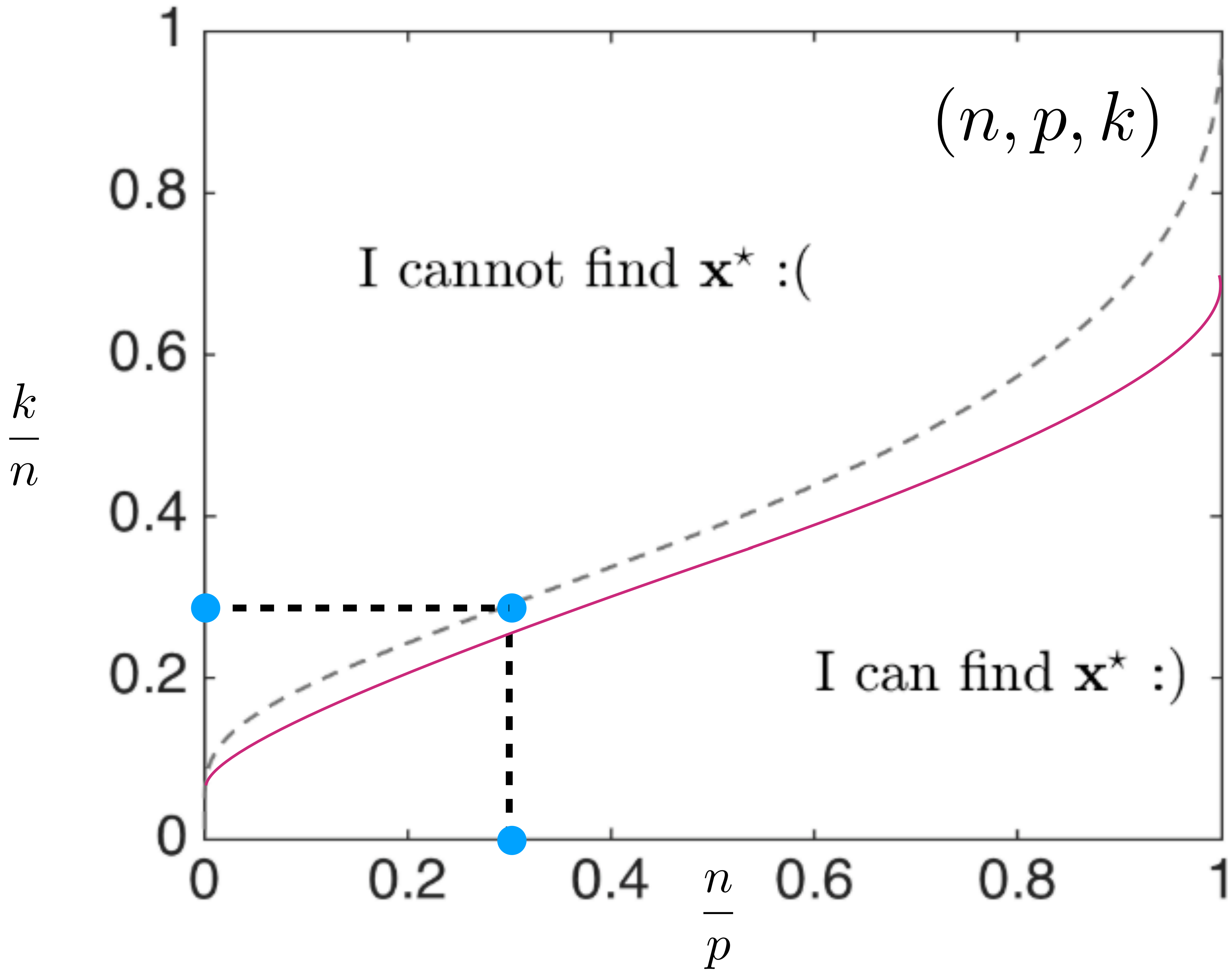
– I.e.,

$$\eta = \arg\min_{\eta \in \mathbb{R}_+} f\left(x_t - \eta \nabla_{Q_t} f(x_t)\right)$$

– Q: "Great! Why don't we use that all the time?"

– A: "Because, moving beyond least squares, solving this might be as difficult as the original problem"

Phase transition update



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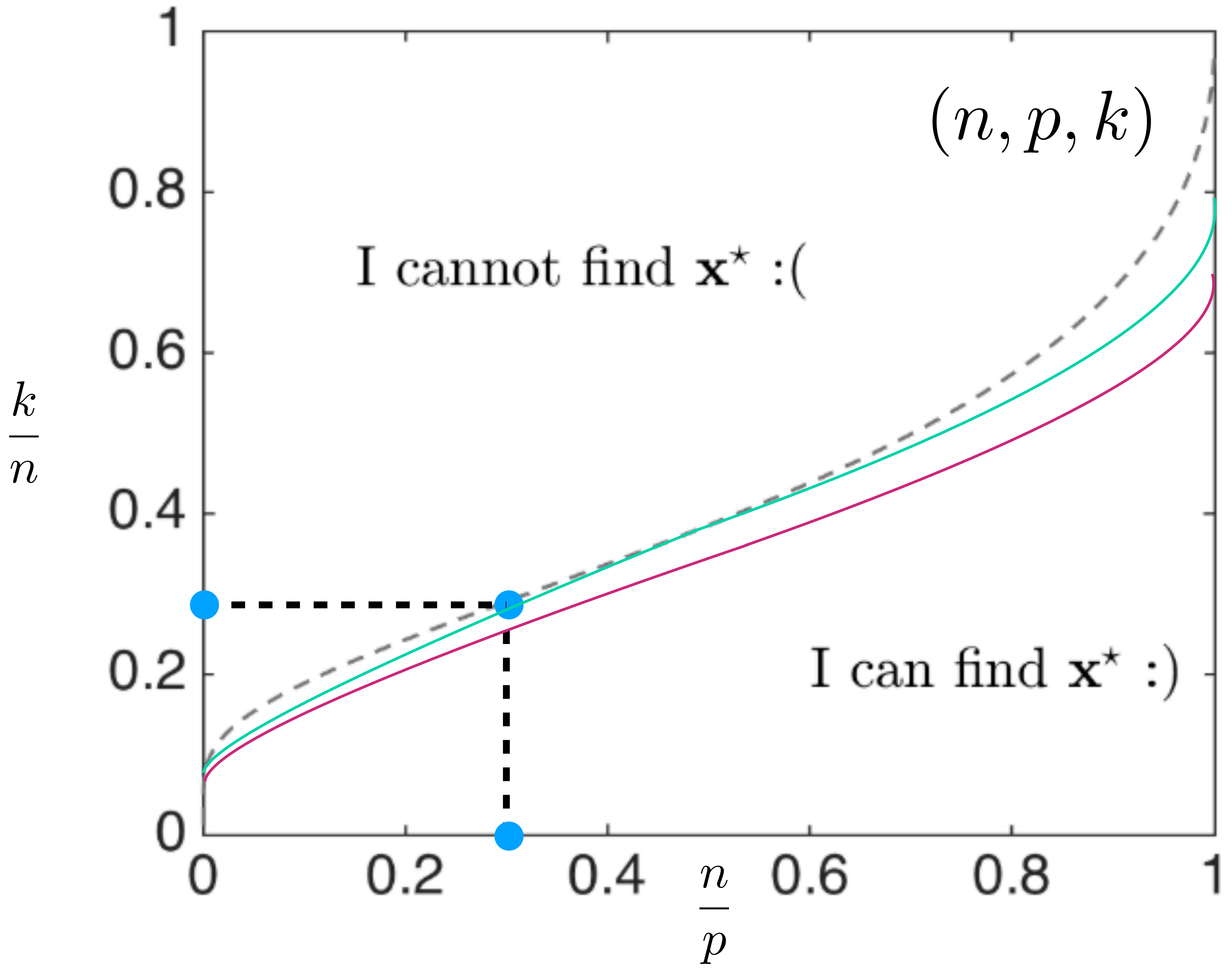

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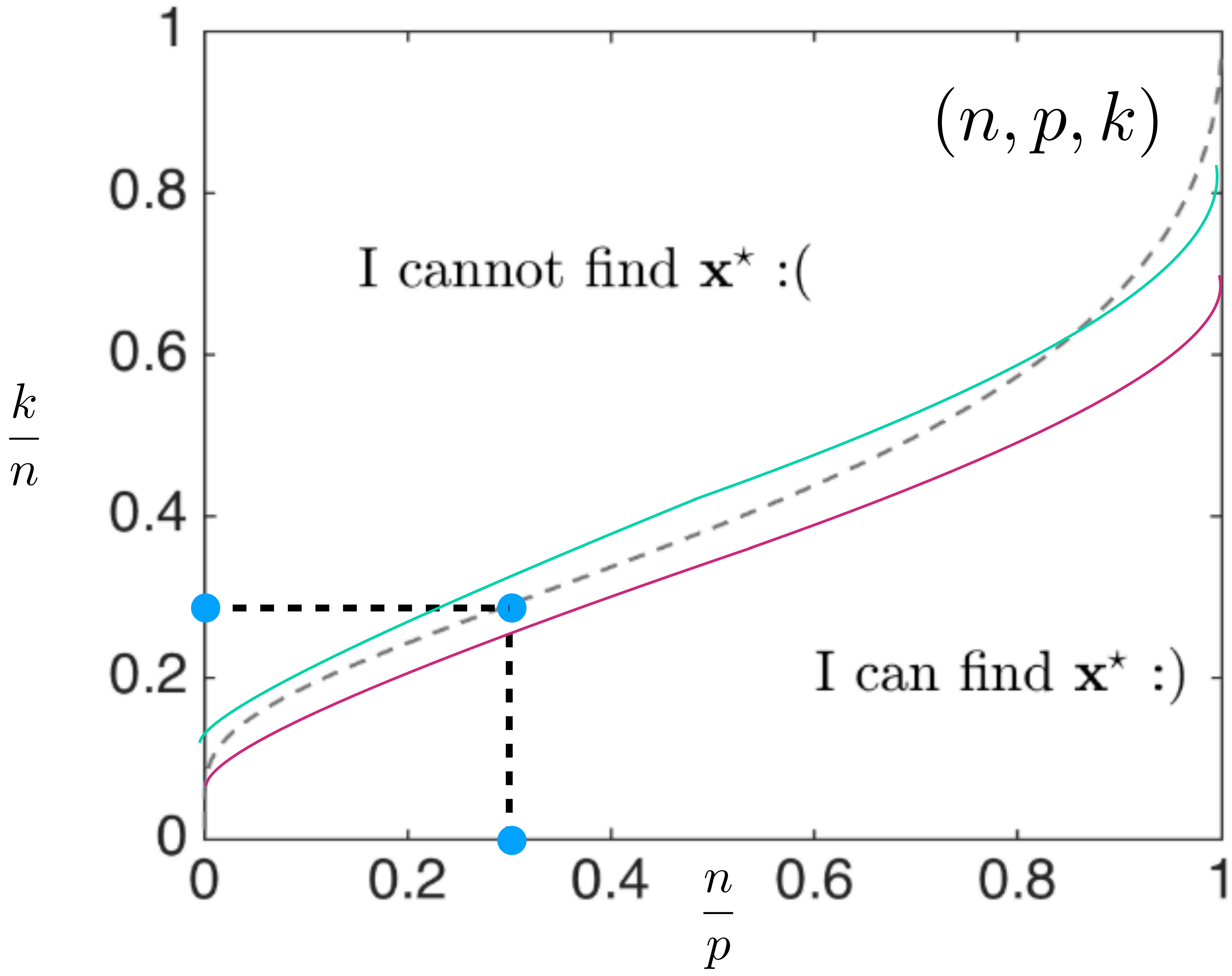

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(Some of these methods can be found in the Review part)

But does this step size selection work in theory?

Whiteboard

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- A: ``We actually get denser and denser solutions``
- Q: ``Is there any non−provable tweaks?``
- A: ``Problem−dependent strategies``

(Demo)

``All these sound interesting.. but do they extend to other objectives? And how are they related with what we discussed so far?``

(Lipschitz gradient continuity, strong convexity, Hessians, etc..)

A different view of RIP

– Reminder

$$(1-\delta)\|x_1 - x_2\|_2^2 \leq \|A(x_1 - x_2)\|_2^2 \leq (1+\delta)\|x_1 - x_2\|_2^2,$$
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- When objective has Lipschitz continuous gradients and is strongly convex:

$$\mu I \preceq \nabla^2 f(x) \preceq LI$$

Restricted smoothness and strong convexity

– ``Restricted``: the properties hold over a subset of $\mathbb{R}^p$

Let's say $x \in \mathcal{C} \subseteq \mathbb{R}^p$

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- Restricted strong convexity

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|_2^2, \quad \forall x, y \in \mathcal{C}$$

(Note that this μ is different from that of general strong convexity)

Examples

– Sparse logistic regression

$\in \{\pm 1\}$

$$\begin{aligned} \min_{x \in \mathbb{R}^p} \quad & \frac{1}{n} \sum_{i=1}^{n} \log\left(1 + \exp\left(-y_i a_i^\top x\right)\right) + \frac{\lambda}{2} \|x\|_2^2 \\ \text{s.t.} \quad & \|x\|_0 \le k \end{aligned}$$

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and restricted smoothness with constant:

$$L = \left(\lambda_{\max}\left(A^\top A, k\right) + \lambda\right)$$

(For more information, see
``Gradient hard thresholding pursuit``)

Examples

- Graphical model selection (under Gaussian assumptions)

Whiteboard

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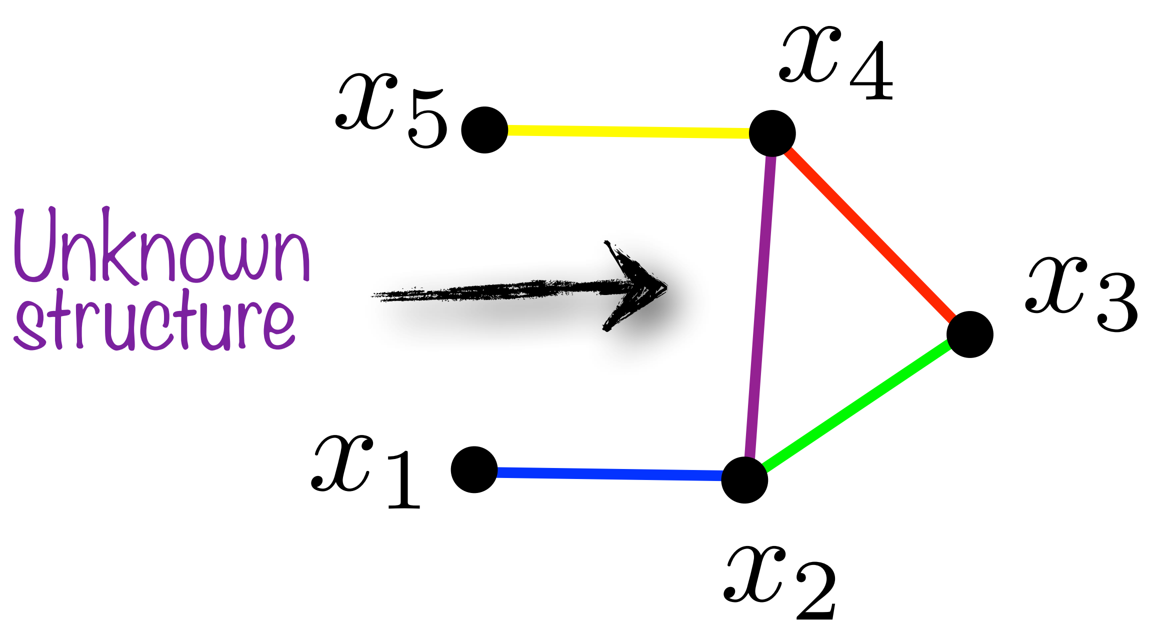

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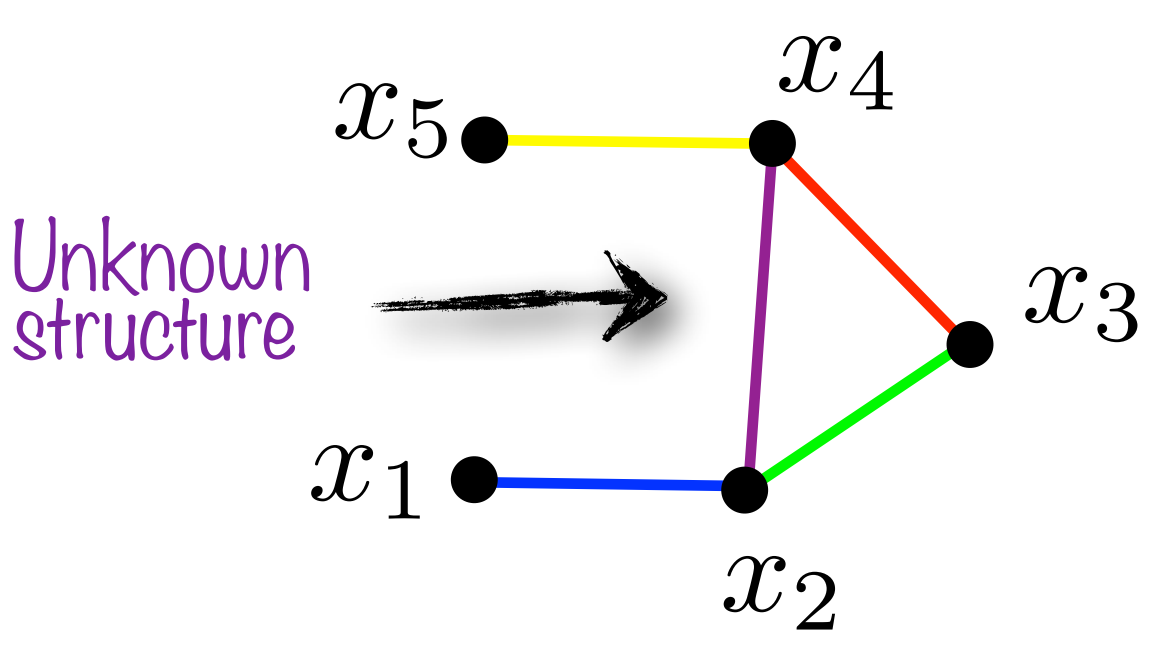
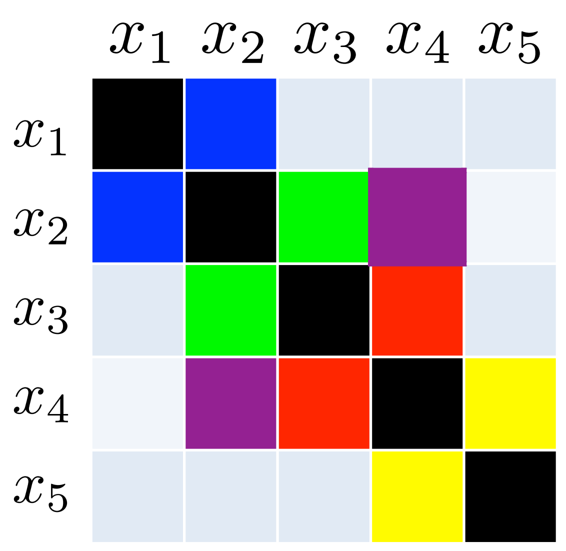


Target

$$\boldsymbol{\Sigma}^{-1} =$$



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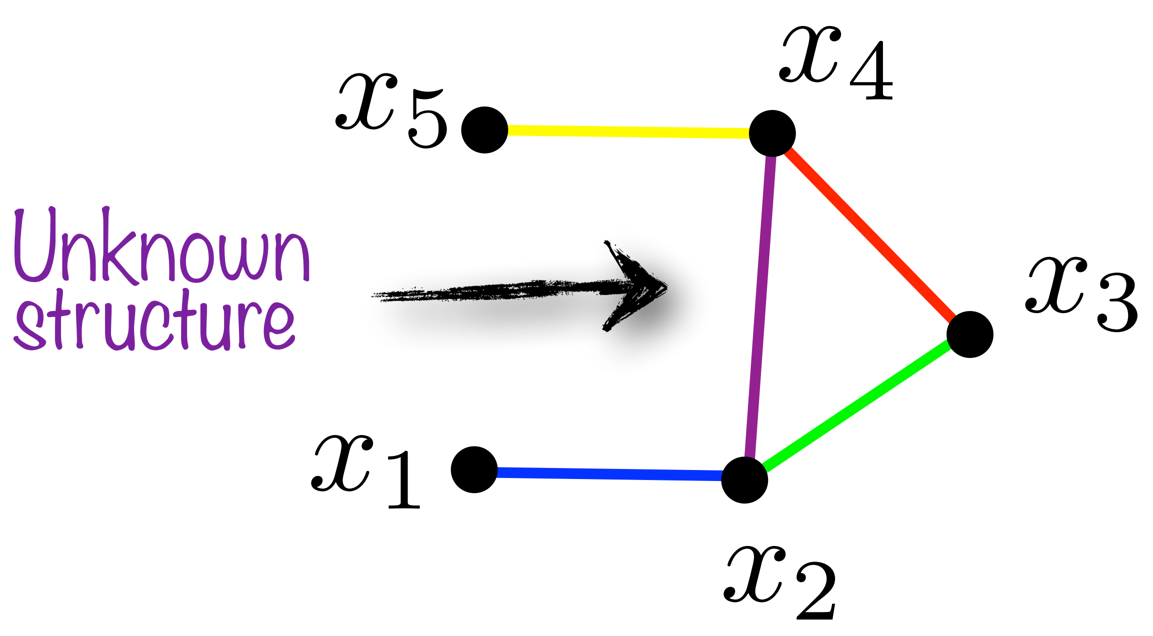

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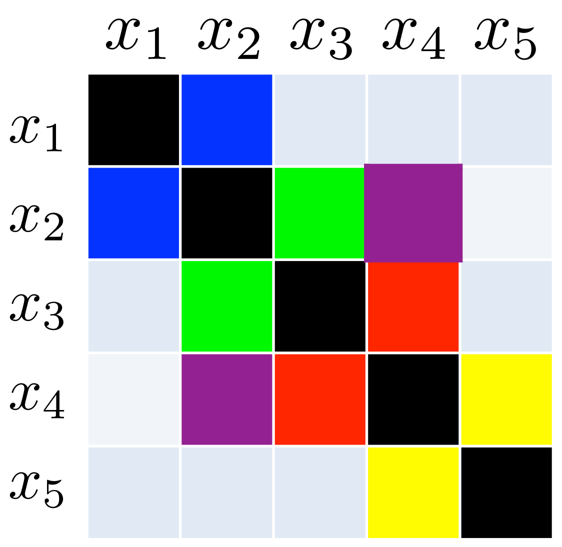
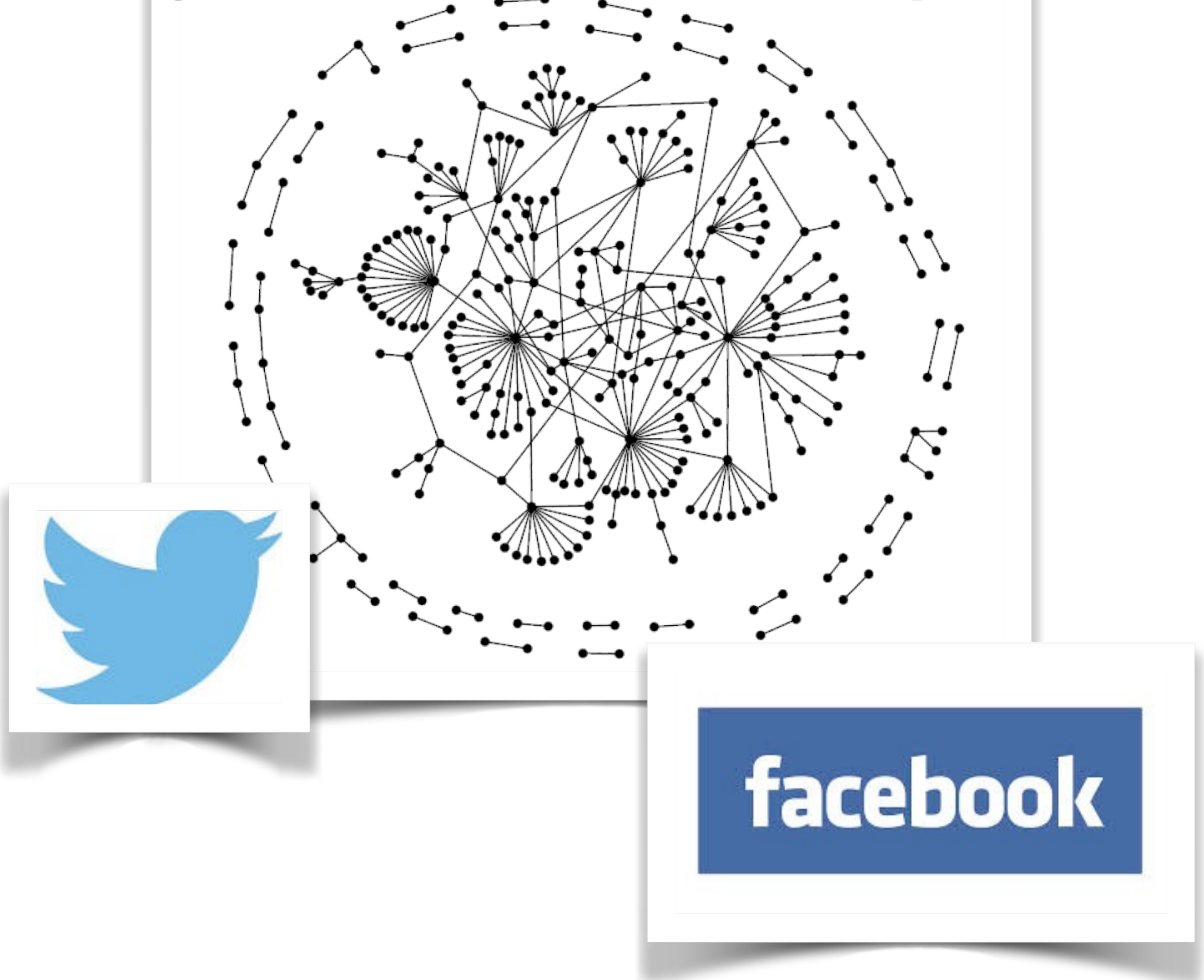


Large-scale real example



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    Input: sample covariance $\widehat{\mathbf{\Sigma}}$ calculated usually from limited samples

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Optimization problem 2

Non-convex

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Beyond plain sparsity

- Our discussion so far holds for discrete structures beyond sparsity:

  Block-sparsity, overlapping block-sparsity, dispersive models, tree sparsity, graph-sparsity, etc..

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- Our discussion so far holds for discrete structures beyond sparsity:

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- As long as the projection onto the combinatorial constraint can be computed efficiently:

$$\begin{aligned} \min_{x \in \mathbb{R}^p} \quad & \|x - y\|_2^2 \\ \text{s.t.} \quad & x \in \mathcal{C} \end{aligned}$$

- Various extensions include inexact projections, greedy approaches, and there are connections with (sub/super)modular optimization

Interlude: Statistics in Data Science

- We will use the example of RIP

- Disclaimer: this is not a complete introduction to concentration inequalities

  (How many would be interested in learning about concentration inequalities (as a course)?)

Conclusion

- This lecture considers **sparse model selection** in Data Science applications
- While there are rigorous and efficient methods also in the convex domain we followed the **non-convex path** of hard thresholding methods
- We discussed some global convergence guarantees, and highlighted the importance of hyper-parameter tuning

Next lecture

- We will consider the case of **low-rank recovery**, natural extension of sparsity - there, we have different ways to exploit non-convexity